

Generalized Scale Invariant Gravity

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We generalize the scale invariant gravity by allowing a negative kinetic energy term for the classical scalar field. This gives birth to a new scalar-tensor theory of gravity, in which the scalar field is in fact an auxiliary field. For a pure gravity theory without matter, the scale symmetric phase represents an equivalent class of gravity theories, which the Einstein gravity plus a cosmological constant belongs to under a special gauge choice. The one-loop quantum correction of the theory is calculated by using the Vilkovisky-DeWitt's method. We find that the scale symmetry is broken dynamically, and that the Einstein gravity is the ground state of the broken phase. We also briefly discuss the consequent cosmological implications. It is shown that the time-delay experiment restricts the present universe to be very close to the ground state.

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I. INTRODUCTION

The classical Einstein gravity, dictated by the symmetry of general transformation of space-time coordinates based on the equivalence principle, agrees well with current observed data. However, it is not unreasonable to ponder other alternatives or modified theories of gravity. Theoretically, the coordinate-reparametrization symmetry is not guaranteed to be the ultimate symmetry of gravity. For example, in contrast with the other three fundamental forces, the quantum theory of Einstein gravity is non-renormalizable and the coordinate-reparametrization symmetry is not enough to cancel all the divergences in quantum corrections. If we believe that physical theories should be self-consistent, symmetries in addition to the reparametrization symmetry should be involved in gravity.

Inspired by the idea of the *relativity of motion* in Einstein's geometrical theory of space-time, Weyl [1] introduced an additional principle, the *relativity of magnitude*, which is realized by the local scale symmetry of space-time. A bonus was that the vector field presented in Weyl gravity looks like the electromagnetic field in Maxwell's theory, so that all known interactions at that time were hopeful to be unified in a geometrical sense. Unfortunately, Weyl's hypothesis had been criticized by Einstein because it implies that the frequency of spectral lines emitted by atoms would not remain constant but would depend on their history. The reason behind is that quantum phenomena provide an absolute standard of scale. An atomic clock measures time in an absolute way and an absolute standard of length is given by taking the light-speed to be unity. So, the scale symmetry seems not to be appreciated by the quantum nature.

This problem can be resolved by either introducing a scale symmetry breaking mechanism [2,3] or following Dirac's suggestion [4]. Dirac supposed that the Einstein equations refer to an space-time interval connecting two neighbouring points which is not the same as the interval measured by atomic apparatus. In this paper we will follow the former wisdom. First of all, we will introduce the classical theory of scale invariant gravity. Then, we will present a generalized form of the theory, and show that it contains a new scalar-tensor theory of gravity. To implement the scale symmetry breaking, we will calculate the one-loop correction of the scalar-tensor action. It will be shown that the quantum effects would indeed break the scale symmetry dynamically. In order to avoid ambiguities arising from field reparametrizations or gauge transformations in doing the quantum corrections, we will adopt the Vilkovisky-DeWitt (VD) [5,6] method to handle the symmetries of general coordinate transformation as well as scale transformation.

The paper is organized as follows. In Section II, we introduce the scale transformation and the classical scale invariant gravity. A new classical theory of the scale invariant gravity is proposed in Section III. In Section IV,

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we calculate the Vilkovisky-DeWitt one-loop effective potential of the theory, with detailed calculations shown in Appendix. In Section V, we briefly discuss the cosmological consequences.

II. CLASSICAL SCALE INVARIANT GRAVITY

Let us begin with a covariant derivative in curved space-time,

$$D_\mu V^\nu = \partial_\mu V^\nu - \Gamma_{\mu\rho}^\nu V^\rho, \quad (1)$$

of a vector field V_μ . As usual, $\Gamma_{\mu\rho}^\nu$ is the Christoffel symbol defined by

$$\Gamma_{\mu\rho}^\nu \equiv \frac{1}{2} g^{\nu\sigma} (g_{\sigma\mu,\rho} + g_{\sigma\rho,\mu} - g_{\mu\rho,\sigma}), \quad (2)$$

where $X_{,\mu}$ denotes $\partial_\mu X$.

Weyl's gauge (scale) transformation means that the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is transformed under the rule,

$$ds^2 \rightarrow ds'^2 = \Omega^2(x) ds^2. \quad (3)$$

If we choose to keep the parametrization of coordinates x^μ invariant, Weyl's gauge transformation can be equivalently represented by the conformal transformation of the metric,

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x). \quad (4)$$

It is clear that Weyl's gauge transformation is not included in Einstein's general transformation of coordinates, under which ds^2 is an invariant. Also, the light-speed still keeps the unity in Weyl theory since Weyl's gauge transformation has nothing to do with null geodesics $ds^2 = 0$. Generally, a quantity X would be said to have conformal weight n if it transforms as

$$X \rightarrow X' = \Omega^n(x) X. \quad (5)$$

Suppose $\Omega(x)$ is a constant of space-time, then the Christoffel symbol would be invariant under this transformation. However, if $\Omega(x)$ is space-time dependent, to keep $\Gamma_{\mu\rho}^\nu$ conformal invariant would require an additional *gauge* field S_μ transformed as

$$S_\mu \rightarrow S'_\mu = S_\mu - \Omega^{-1}(x) \partial_\mu \Omega(x), \quad (6)$$

and the partial derivative in the Christoffel symbol replaced by the conformal covariant derivative

$$\mathcal{D}_\mu g_{\nu\rho} \equiv (\partial_\mu + 2S_\mu) g_{\nu\rho}. \quad (7)$$

Hence the conformal invariant affine connection is given by

$$\begin{aligned} \tilde{\Gamma}_{\mu\rho}^\nu &\equiv \frac{1}{2} g^{\nu\sigma} (\mathcal{D}_\rho g_{\sigma\mu} + \mathcal{D}_\mu g_{\sigma\rho} - \mathcal{D}_\sigma g_{\mu\rho}) \\ &= \Gamma_{\mu\rho}^\nu + \delta_\mu^\nu S_\rho + \delta_\rho^\nu S_\mu - g_{\mu\rho} S^\nu, \end{aligned} \quad (8)$$

and the corresponding scalar curvature becomes

$$\tilde{R} = R + 6D_\mu S^\mu + 6S_\mu S^\mu. \quad (9)$$

Note that $\mathcal{D}_\mu g_{\nu\rho}$ and \tilde{R} have weights 2 and -2 respectively. Furthermore, the conformal covariant derivative of a vector field V^ν with weight n is written as

$$\mathcal{D}_\mu V^\nu \equiv (\partial_\mu + nS_\mu) V^\nu - \tilde{\Gamma}_{\mu\rho}^\nu V^\rho. \quad (10)$$

It immediately follows that all parameters in any gravity theory with scale invariance are dimensionless. In order to have Einstein gravity as the effective theory at low energies, the dimensional "constants" in the Einstein gravity such as the gravitational constant and the cosmological constant must correspond to quantities composed of some fields with dimension, whose values may also be changing with space-time. To implement this, we introduce an additional

scalar field $\hat{\phi}$ with weight -1 . A self-interaction term of $\hat{\phi}$, $V(\hat{\phi})$, can also be added as long as it is being scale invariant:

$$\frac{\delta}{\delta\Omega}\sqrt{-g'}V(\hat{\phi}') = \sqrt{-g'}\Omega^{-1}\left[4V(\hat{\phi}') - \hat{\phi}'\frac{\delta V(\hat{\phi}')}{\delta\hat{\phi}'}\right] = 0, \quad (11)$$

which simply means that the stress-energy tensor of $\hat{\phi}$ must be traceless. The only non-trivial choice of V consistent with the above equation is

$$V(\hat{\phi}) = \frac{\lambda}{4!}\hat{\phi}^4, \quad (12)$$

up to a coupling constant λ .

Thus, we can construct a simple scale invariant action for gravity as

$$S_{(\xi>0)} = \int d^4x \sqrt{-g} \left[-\frac{1}{2}\xi\hat{\phi}^2\tilde{R} - \frac{1}{2}\mathcal{D}_\mu\hat{\phi}\mathcal{D}^\mu\hat{\phi} - V(\hat{\phi}) - \frac{1}{4}H_{\mu\nu}H^{\mu\nu} \right], \quad (13)$$

in which the signature is $(-, +, +, +)$, and

$$\tilde{R} = R + 6f(D_\mu + fS_\mu)S^\mu, \quad (14)$$

$$\mathcal{D}_\mu\hat{\phi} = (\partial_\mu - fS_\mu)\hat{\phi}, \quad (15)$$

$$H_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu, \quad (16)$$

where f is a coupling constant, and $\xi > 0$ to allow for a positive gravitational constant. The action is invariant under the local scale transformations,

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x), \quad (17)$$

$$\hat{\phi}(x) \rightarrow \hat{\phi}'(x) = \Omega^{-1}(x)\hat{\phi}(x), \quad (18)$$

$$S_\mu(x) \rightarrow S'_\mu(x) = S_\mu(x) - f^{-1}\Omega^{-1}(x)\partial_\mu\Omega(x). \quad (19)$$

In fact, this action is an usual form of the so-called classical scale invariant gravity theory which has been extensively studied in the literatures [4,7].

III. GENERALIZED SCALE INVARIANT GRAVITY

Before we study the mass spectrum of the Lagrangian in the action $S_{(\xi>0)}$ (13), let us rewrite it in a more convenient form,

$$L_{(\xi>0)} = -\sqrt{-g} \left[\frac{1}{2}\xi\hat{\phi}^2R + \frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} + \frac{1}{2}(1+6\xi)f\hat{\phi}^2(fS_\mu S^\mu + D_\mu S^\mu) + \frac{1}{4}H_{\mu\nu}H^{\mu\nu} + V(\hat{\phi}) \right]. \quad (20)$$

Suppose the true vacuum state be the Minkowski flat space-time obeying Einstein gravity, then we perturb the classical fields about this vacuum,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (21)$$

$$\frac{1}{2}\xi\hat{\phi}^2 = \frac{1}{16\pi G}e^{\alpha\sigma}, \quad (22)$$

$$S_\mu = s_\mu, \quad (23)$$

where α is an arbitrary parameter, and G is the Newton's constant. Also, all $h_{\mu\nu}$, σ , and s_μ are small perturbations. After redefining a new metric perturbation,

$$\rho_{\mu\nu} = h_{\mu\nu} + \alpha\sigma\eta_{\mu\nu}, \quad (24)$$

the kinetic part in $L_{(\xi>0)}$ can be expanded up to quadratic terms into

$$\begin{aligned}\mathcal{L}_{K.E.} = & -\frac{1}{16\pi G} \left\{ \frac{1}{4} \partial_\mu \rho_{\alpha\beta} \partial^\mu \rho^{\alpha\beta} - \frac{1}{4} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} \partial_\alpha \rho^{\alpha\beta} \partial_\beta \rho - \frac{1}{2} \partial_\alpha \rho^{\alpha\mu} \partial^\beta \rho_{\beta\mu} \right. \\ & \left. + \frac{1+6\xi}{\xi} \left[\frac{\alpha^2}{4} \partial_\mu \sigma \partial^\mu \sigma + f^2 s_\mu s^\mu + f \partial_\mu s^\mu \right] \right\} - \frac{1}{4} (\partial_\mu s_\nu - \partial_\nu s_\mu) (\partial^\mu s^\nu - \partial^\nu s^\mu),\end{aligned}\quad (25)$$

where $\rho = \eta^{\mu\nu} \rho_{\mu\nu}$. To maintain the stability of the theory, one has to assure the positivity of the kinetic energy as well as the mass of each perturbation. This requires that $(1+6\xi)/\xi$ must be greater than zero, i.e., either $\xi > 0$ or $\xi \leq -1/6$. We thus see that the stability condition does not rule out a negative value for ξ . However, that $\xi \leq -1/6$ would imply an unwanted negative gravitational constant. Therefore, in order to include a negative ξ , we propose the following generalized scale invariant action for gravity,

$$S_\xi = \int d^4x \sqrt{-g} \left[-\frac{1}{2} |\xi| \hat{\phi}^2 \tilde{R} - \frac{|\xi|}{\xi} \left(\frac{1}{2} \mathcal{D}_\mu \hat{\phi} \mathcal{D}^\mu \hat{\phi} + V(\hat{\phi}) \right) - \frac{1}{4} H_{\mu\nu} H^{\mu\nu} \right], \quad (26)$$

where ξ is any real number excluding the interval $(-1/6, 0]$. Obviously, this action reduces to $S_{(\xi>0)}$ when $\xi > 0$. Note that the kinetic energy of $\hat{\phi}$ in the action (26) would be negative when $\xi \leq -1/6$. This may bring out an energy crisis at the classical level, even though we have just shown that the positivity of energy is guaranteed in the spectrum of the theory. However, we will show in Section V that the total energy of the classical theory is bounded below and well-defined.

When $\xi = -1/6$, the gauge field S_μ in the action (26) becomes massless and decouples from the scalar field $\hat{\phi}$, namely,

$$S = S_{grav} + S_{gauge}; \quad (27)$$

$$S_{grav} = \int d^4x \sqrt{-g} \left[-\frac{1}{12} \hat{\phi}^2 R + \frac{1}{2} \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} + V(\hat{\phi}) \right], \quad (28)$$

$$S_{gauge} = \int d^4x \sqrt{-g} \left[-\frac{1}{4} H_{\mu\nu} H^{\mu\nu} \right]. \quad (29)$$

In this case the resulting action S_{grav} describes a theory of a scalar field $\hat{\phi}$ conformally coupled to gravity, which is invariant under the local scale transformations (17) and (18). Note that S_{gauge} is by itself invariant under the transformation (19) with the transformation function $\Omega(x)$ independent of the ones for S_{grav} . Hence the dynamical degrees of freedom (DOF) of the $\xi = -1/6$ theory is less than those of the cases with $\xi \neq -1/6$ by one. For $\xi \neq -1/6$, the dynamical DOF is nine: ten DOF for graviton, one for scalar field and four for gauge field altogether amount to fifteen DOF, which is then deduced by one scale symmetry, four coordinate reparametrization symmetry and a constraint Gauss' law. And for $\xi = -1/6$, the dynamical DOF is eight. Also, from Eq. (25), we see that $\hat{\phi}$ is in fact an auxiliary field in the spectrum. This is a manifestation of the scale symmetry. To fix the degree of freedom corresponding to the scale symmetry, one can put a constraint, the *Einstein gauge*,

$$\hat{\phi} = v, \quad (30)$$

with v being a non-zero constant, or equivalently,

$$\Omega^2(x) = v^{-2} \hat{\phi}^2, \quad (31)$$

so that the action S_{grav} becomes

$$S_{EG} = \int d^4x \sqrt{-g} \left[-\frac{1}{16\pi G} R + 2\Lambda \right], \quad (32)$$

which is identical to the Einstein gravity with the gravitational constant $G = 3/(4\pi v^2)$ and the cosmological constant $\Lambda = \lambda v^4/48$. We thus see that the so-called Weyl gravity (28) is no more than a generalized form of the Einstein gravity S_{EG} . In general, it represents a set of pure gravity theories with arbitrary space-time varying gravitational and cosmological "constants" interconnected by the local scale transformation.

Below we will restrict our calculations to the massless case (28) for simplicity. Technically, this suffices to show how we would deal with the local scale symmetry in doing quantum corrections in the next section. The method can be equally well applied to the massive cases. It is rather interesting to see how the massless case can be related to the massive cases through a running coupling constant ξ .

IV. ONE-LOOP EFFECTIVE POTENTIAL

To study the quantum aspect of the theory (28), it is convenient to start with the loop-corrected effective action [8, 9]. Since we are mainly interested in using the quantum corrected result to study the ground state of the theory and also their implications to classical cosmology, we are going to ignore the singular behavior of the system. Thus it is appropriate to use the perturbation theory, in which the gravity is depicted as a theory of massless spin-two gauge particles, namely, the gravitons. However, there exist some ambiguities when one applies the conventional Coleman-Weinberg formalism to gauge theories.

The first ambiguity is that the conventional effective action is not invariant under field reparametrizations or, in particular, gauge transformation. Although technically the gauge invariance of the effective action can be achieved by splitting each field into quantum- and background-part, and making an arrangement so that the gauge of the quantum-field-part is fixed while the effective action composed by the background-field-part is gauge-invariant [10], it is not guaranteed that the same effective action can be obtained if one initially chooses another gauge condition or another field parametrization for quantum fields.

Even if a gauge invariant effective potential is worked out, usually it still remains a dependence on the coupling factor of the gauge fixing term. This factor gives rise to another ambiguity, especially in direct applications of the effective potential in practical situations. Fortunately, the whole problem was resolved by Vilkovisky [5] and DeWitt [6]. Recalling Einstein's idea in formulating the theory of general relativity: the coordinate reparametrization dependence can be eliminated by simply choosing all variables and derivatives to be covariant, Vilkovisky and DeWitt pointed out that the space of the field configurations may not be trivially flat. Thus, in order to obtain an effective action independent to the field reparametrization, one should properly construct covariant variables and derivatives in the configuration space. Furthermore, it is possible to define a gauge invariant metric in the configuration space, so that a gauge independent effective potential can be uniquely derived without any ambiguity.

As the effective potential is taken from the zeroth order of the momentum expansion of the effective action, the effective potential is off-shell except at the vacuum, where the field configuration satisfies the equation of motion. So, it is not unexpected that the shape of the effective potential is gauge-dependent. However, the order of the phase transition during the symmetry breaking process depends on the shape of the effective potential, and a gauge-independent effective potential is needed in computing thermal quantities sensitive to the order of phase transition. In addition, a gauge-independent effective potential is useful in some cases in which the existence of the imaginary part of the effective potential depends on gauge choices [11].

As such, we will adopt the Vilkovisky and DeWitt's (VD) method to calculate the loop corrections. Instead of displaying the full VD calculation of the effective potential, we begin with the conventional Coleman-Weinberg formalism and modify it to the VD calculations when necessary. A brief account of the VD method and a detailed calculation of the effective potential are given in Appendix A.

Expanding the gravitational field and the scalar field about the ground-state background, one has

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi^{-1} h_{\mu\nu}, \quad (33)$$

$$\hat{\phi} = \phi + \sigma, \quad (34)$$

where $\eta_{\mu\nu}$ is a flat metric, ϕ is a constant field, and the quantum field $h_{\mu\nu}$ is graviton. In order to make the generators of the transformation of the scalar field dimensionless, one may define

$$\tilde{\phi} \equiv \phi^{-1} \hat{\phi} = 1 + \phi^{-1} \sigma. \quad (35)$$

To evaluate the one-loop effective potential, it suffices to expand the action (28) up to the second order in the quantum fields, namely,

$$\begin{aligned} -\sqrt{g}\mathcal{L} \approx \mathcal{L}_2 \equiv & V(\phi) + \frac{V(\phi)}{2\phi} h + V'(\phi) \sigma \\ & - \frac{1}{24} \left[h^{\mu\nu}{}_{,\mu} h_{,\nu} - \frac{1}{2} h_{,\mu} h^{\mu}{}_{,\nu} - h^{\mu\nu}{}_{,\nu} h_{\mu\rho}{}^{,\rho} + \frac{1}{2} h^{\mu\nu,\rho} h_{\mu\nu,\rho} - 4\sigma(h_{,\mu}{}^{,\mu} - h^{\mu\nu}{}_{,\mu\nu}) \right] \\ & + \frac{1}{2} \sigma_{,\mu} \sigma^{,\mu} + \frac{1}{8} \frac{V}{\phi^2} h^2 + \frac{1}{2} \frac{V'}{\phi} h \sigma + \frac{1}{2} V'' \sigma^2 - \frac{1}{4} \frac{V}{\phi^2} h^{\mu\nu} h_{\mu\nu}, \end{aligned} \quad (36)$$

where $h \equiv h_{\mu}{}^{\mu}$ and $V' \equiv \delta V / \delta \phi$. One may write the quadratic part of the Lagrangian in the following form,

$$\mathcal{L}_q = \frac{1}{2} \psi_a P^{ab} \psi_b, \quad (37)$$

where $a, b = 0, \dots, 10$ and ψ_a represent the quantum fields σ and ten independent components of $h_{\mu\nu}$ respectively.

Suppose a transformation

$$\rho_{\mu\nu} = h_{\mu\nu} + 2\sigma\eta_{\mu\nu} \quad (38)$$

is performed, the quadratic Lagrangian becomes

$$\begin{aligned} \mathcal{L}_2 = & -\frac{1}{24} \left(\frac{1}{2} \rho \partial^2 \rho + \rho_{\mu\nu} \partial^\nu \partial_\beta \rho^{\mu\beta} - \rho \partial_\mu \partial_\nu \rho^{\mu\nu} - \frac{1}{2} \rho_{\mu\nu} \partial^2 \rho^{\mu\nu} \right) + \frac{V}{4\phi^2} \left(\frac{1}{2} \rho^2 - \rho_{\mu\nu} \rho^{\mu\nu} \right) \\ & + \sigma A \rho + \sigma B \sigma, \\ A \equiv & \frac{V'}{2\phi} - \frac{V}{\phi^2}, \\ B \equiv & \frac{V''}{2} - \frac{4V'}{\phi} + \frac{4V}{\phi^2}, \end{aligned} \quad (39)$$

which is equivalent to the Lagrangian (25) with $\xi = -1/6$, where we have performed a different expansion scheme in order to make the gravitational constant G explicit. Here the kinetic term of σ field vanishes, that is, σ is an auxiliary field. This gives a hint that there are some symmetries in this Lagrangian. Indeed, they originate from the scale transformations (17) and (18). When the Einstein gauge (31) is chosen, the gravitational field becomes

$$g'_{\mu\nu} = v^{-2} \hat{\phi}^2 g_{\mu\nu}. \quad (40)$$

Substituting the background field expansions (33) and (34) into the above transformation, one would obtain the infinitesimal version of the scale transformation,

$$\rho_{\mu\nu} \equiv h'_{\mu\nu} = h_{\mu\nu} + 2\sigma\eta_{\mu\nu} + O(\psi^2), \quad (41)$$

if $v = \phi$. This is exactly the transformation (38).

One can go further by letting

$$\sigma' = \sigma + \frac{A}{2B} \rho = \left(1 + \frac{4A}{B} \right) \sigma + \frac{A}{2B} h, \quad (42)$$

so that the Jacobian with respect to the field reparametrization $(h_{\mu\nu}, \sigma) \rightarrow (\rho_{\mu\nu}, \sigma')$ equals unity. Then the last two terms in Eq. (39) turn into

$$\sigma A \rho + \sigma B \sigma = \rho \left(\frac{A^2}{4B} \right) \rho + \sigma' B \sigma', \quad (43)$$

where σ' is now decoupled. The field equation $\sigma' = 0$ corresponds to the Einstein gauge (30). We thus conclude that the only difference between the Einstein gravity and the Weyl gravity (28) is that the latter has the gravitational wave with a mass term proportional to ρ^2 as shown in the right-hand side of Eq. (43). Nevertheless, this difference can be lifted by choosing a traceless gauge $\rho = 0$.

In fact, after the Lagrangian (37) being diagonalized, there are other vanishing kinetic terms, which correspond to four degrees of freedom of the reparametrization symmetry. This property is not surprising in systems with internal symmetries. Consider a symmetry transformation operator U operating on a quantum field by $\psi_a \rightarrow \psi'_a = U_a{}^b \psi_b$. Assume the quadratic Lagrangian is invariant under such a symmetry transformation, then $P = U^T P U$ where U^T denotes the transverse of U . This would imply that $\det P = 0$ if $\det U \neq 1$. In our model (36), the determinants of the operators corresponding to the scale transformation and reparametrization are indeed not equal to unity. As such, the corresponding operator P^{ab} ($V = 0$) is not invertible, and hence its propagator cannot be defined. Similar situations occur in the theory of electromagnetism, where one can interpret P as a projection operator [12].

It is therefore preferable to set the gauge condition in a form of a first derivative with respect to the corresponding quantum field in this case. For example, in electromagnetism, one may choose the Lorentz gauge $\partial_\mu A^\mu = 0$, whose corresponding gauge fixing term reads

$$\frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \quad (44)$$

with an arbitrary factor α . Once this term is added into the Lagrangian of electromagnetism, the propagator of photon is then well-defined up to the factor α .

In the present case, the reparametrization symmetry can be fixed by choosing

$$h_{\mu\nu,}{}^\nu - \frac{1}{2}h_{,\mu} = 0, \quad (45)$$

while the gauge condition for the scale symmetry (30) is equivalent to

$$\partial_\mu \sigma = 0. \quad (46)$$

Hence, the gauge fixing term then reads

$$\mathcal{L}_{gf} = \frac{1}{2\alpha}(h_{\mu\rho,}{}^\mu - \frac{1}{2}h_{,\rho})(h^{\nu\rho,}{}_\nu - \frac{1}{2}h_{,\rho}) + \frac{1}{2\beta}\sigma_{,\mu}\sigma^{,\mu}, \quad (47)$$

where α and β are arbitrary factors.

In the one-loop level, only quadratic terms are needed in path-integral calculation. Since the corresponding ghost Lagrangian,

$$\mathcal{L}_{ghost} = \bar{\eta}_\mu(-\partial^2)\eta^\mu, \quad (48)$$

is totally decoupled from the system in this level, the ghost field can be neglected here. The quadratic part of the Lagrangian considered in this approximation is

$$\begin{aligned} \mathcal{L}_q = & \frac{1}{4}h_{\mu\nu}\alpha_1 h^{\mu\nu} - \frac{1}{4}h\alpha_2 h + \frac{1}{2}h_{\mu\nu}\alpha_3 \frac{\partial^\nu \partial_\rho}{\partial^2} h^{\mu\rho} - \frac{1}{2}h\alpha_4 \frac{\partial_\mu \partial_\nu}{\partial^2} h^{\mu\nu} - \frac{1}{2}h_{\mu\nu}\alpha_5 \frac{\partial^\mu \partial^\nu \partial_\rho \partial_\sigma}{\partial^4} h^{\rho\sigma} \\ & + \frac{1}{2}\sigma\beta_1\sigma + \frac{1}{2}\sigma\beta_2 h + \frac{1}{2}\sigma\beta_3 \frac{\partial_\mu \partial_\nu}{\partial^2} h^{\mu\nu}, \end{aligned} \quad (49)$$

where

$$\begin{aligned} \alpha_1 &= -\frac{k^2}{12} - \frac{\lambda\phi^2}{24}, \\ \alpha_2 &= -\left(\frac{1}{12} + \frac{1}{2\alpha}\right)k^2 - \frac{\lambda\phi^2}{48}, \\ \alpha_3 &= \alpha_4 = \left(\frac{1}{12} + \frac{1}{\alpha}\right)k^2, \\ \alpha_5 &= 0, \\ \beta_1 &= \left(1 + \frac{1}{\beta}\right)k^2 + \frac{\lambda\phi^2}{2}, \\ \beta_2 &= \frac{k^2}{3} + \frac{\lambda\phi^2}{6}, \\ \beta_3 &= -\frac{k^2}{3}, \end{aligned} \quad (50)$$

with a replacement $-\partial^2 \rightarrow k^2$. Here ∂^4 is a shorthand of $(\partial_\mu \partial^\mu)^2$. Let us write \mathcal{L}_q in the form of Eq. (37), the eigenvalues of P^{ab} are found to be

$$\begin{aligned} \lambda_1 &= \lambda_2 = \lambda_3 = -\frac{k^2}{12} - \frac{\lambda\phi^2}{24}, \\ \lambda_4 &= \lambda_5 = \lambda_6 = \frac{k^2}{\alpha} - \frac{\lambda\phi^2}{24}, \\ \lambda_7 &= \lambda_8 = -\frac{k^2}{24} - \frac{\lambda\phi^2}{48}, \\ \lambda_0\lambda_9\lambda_{10} &= \frac{k^6}{48\alpha\beta} + \lambda\phi^2 \left(\frac{1}{96\alpha\beta} - \frac{1}{48\alpha} - \frac{1}{1152\beta} \right) k^4 \\ &+ (\lambda\phi^2)^2 \left(\frac{1}{3456} - \frac{1}{64\alpha} - \frac{1}{2304\beta} \right) k^2 + \frac{5}{13824} (\lambda\phi^2)^3. \end{aligned} \quad (51)$$

Indeed, if $1/\alpha$ and $1/\beta$ are set to be zero such that \mathcal{L}_{gf} vanishes, there would be five elements of the eigenvector which have no kinetic terms. They are $\lambda_4, \lambda_5, \lambda_6$ and two of the three eigenvalues λ_0, λ_9 and λ_{10} .

In terms of λ 's, the unrenormalized one-loop effective potential can be written as

$$V_1 = V - \frac{i}{2} \sum_{a=0}^{10} \text{Tr} \ln \lambda_a. \quad (52)$$

Obviously, the conventional effective potential obtained by substituting the eigenvalues (51) into Eq. (52) depends on arbitrary factors α and β . To eliminate this ambiguity, one should introduce the Vilkovisky-DeWitt effective potential.

From Appendix A, the Vilkovisky-DeWitt method changes the eigenvalues into

$$\begin{aligned} \lambda_1 = \lambda_2 = \lambda_3 &= -\frac{k^2}{12} - \frac{\lambda\phi^2}{24}, \\ \lambda_4 = \lambda_5 = \lambda_6 &= \frac{k^2}{\alpha}, \\ \lambda_7 = \lambda_8 &= -\frac{k^2}{24} - \frac{\lambda\phi^2}{48}, \\ \lambda_0 \lambda_9 \lambda_{10} &= \frac{k^4}{48\alpha\beta} \left(k^2 + \frac{505}{1058} \lambda\phi^2 \right), \end{aligned} \quad (53)$$

by combining the original quadratic Lagrangian (49) with the correction (A44). The one-loop VD effective potential then reads

$$V_1^{VD} = \frac{\lambda\phi^4}{4!} + \frac{5i}{2} \text{Tr} \ln \left(k^2 + \frac{1}{2} \lambda\phi^2 \right) + \frac{i}{2} \text{Tr} \ln \left(k^2 + \frac{505}{1058} \lambda\phi^2 \right) + \text{constant}. \quad (54)$$

This result is similar to those obtained from the simple massless ϕ^4 theory. Following the standard renormalization process [12], the renormalized VD effective potential can be calculated as

$$V_1^{VD} = \frac{\lambda\phi^4}{4!} - \frac{(\lambda\phi^2)^2}{64\pi^2} K \left(\ln \frac{\phi^2}{M} - \frac{25}{6} \right) + \Lambda + O(\lambda^3), \quad (55)$$

$$K = \frac{5}{4} + \left(\frac{505}{1058} \right)^2, \quad (56)$$

where M is a scaling factor, and Λ is a renormalized constant which will be determined in the following section. If we simply choose the Landau-DeWitt gauge, $\alpha = \beta = 0$, in the conventional effective potential obtained from the eigenvalues (51), we would obtain a slightly different value of $K = 3/2$.

Note that the renormalized effective potential (55) would have no imaginary part if λ is positive. In contrary, in the massive case (13) [3], the imaginary part of the conventional effective potential under the Landau-DeWitt gauge is nonzero as long as $\lambda > 0$, while it vanishes when $\lambda < 0$. Moreover, some other gauge choices may make the imaginary part always exist. Since λ has to be positive in order to allow for a stable vacuum, our results suggest that the coupling constant $\lambda|\xi|/\xi$ of the self-interaction in the action (26) should be always negative.

Here we have some remarks and comments on the VD method. First, it is obvious that there is no special values of α and β be chosen such that $V_1 = V_1^{VD}$ in this scale invariant gravity. Second, the effect of VD method on the eigenvalues λ_4, λ_5 and λ_6 is to remove the vertex term $-\lambda\phi^2/24$ (see Eqs. (51) and (53)), so that the loops of ψ_4, ψ_5 and ψ_6 are decoupled from the system. This decoupling can also be achieved by naively choosing the Landau-DeWitt gauge. However, it is not the case in calculating $\lambda_0 \lambda_9 \lambda_{10}$. The complicated mixing between these quantum fields makes the VD residual non-zero vertex term different from that obtained from the Landau-DeWitt gauge. In general, the equality between the naive and the VD effective potentials occurs only in some special cases by accident.

Recall that when a constrained or gauge system is quantized in the path integral formalism, the constraints or gauge conditions $\delta(F[\phi])$ are loosed into a Gaussian distribution $e^{-F^2/2\alpha}$, where α can be defined as the width of this distribution. In other words, all of the off-shell field configurations, which are weighted by a Gaussian, are taken into account, and the constraint $F[\phi] = 0$ is true only when the system is on-shell. Therefore, to choose the Landau-DeWitt gauge, $\alpha \rightarrow 0$, is equivalent to narrowing the Gaussian distribution to a delta-function-like distribution. However, if the configuration space is curved, the direction that $\alpha \rightarrow 0$ may not be orthogonal to the on-shell surface $F[\phi] = 0$.

everywhere because α is not a covariant quantity in \mathcal{M} . Hence taking the Landau-DeWitt gauge naively without considering the curvature effect may give the wrong result.

For example, if we choose the Landau-DeWitt gauge in the conventional effective potential obtained from Eq. (51), the resulting effective potential is identical to the conventional effective potential obtained from Einstein gravity (32) with $\alpha \rightarrow 0$. It should be emphasized that the Einstein gravity is the consequence of choosing the gauge, $\sigma = 0$, in the scale invariant gravity (39) *before* quantization. However, neither of them is equal to the VD effective potential (54). Although the Einstein gravity and the scale invariant gravity are classically equivalent, the off-shell structure as well as the quantum theory of them are quite different.

The third remark is that the field σ , which has no kinetic term, can be identified as an auxiliary field in the classical theory. One may substitute the equation of motion with respect to σ in the Lagrangian to eliminate the auxiliary field and get a new pure graviton theory. If we include only the graviton gauge-fixing α -term to compute the one-loop VD effective potential for this new theory, the results would depend on α . This is because we have ignored the scale symmetry hidden in the new theory. Therefore, if one finds that the obtained VD effective potential still depends on some arbitrary factor which corresponds to the known symmetry of a system, then the system would have to carry some extra hidden symmetry. In this case, one may apply the method developed by Dirac [13] to find all the constraints and then run the quantization process again.

V. VACUUM EXPECTATION VALUE

The vacuum expectation value of ϕ is located at the minimum of V_1^{VD} , namely,

$$\left. \frac{\delta V_1^{VD}}{\delta \phi} \right|_{\langle \phi \rangle} = 0. \quad (57)$$

From Eq. (55), this implies that

$$\langle \phi \rangle = \sqrt{M} \exp \left[-\frac{4\pi^2}{3\lambda K} + \frac{11}{6} \right]. \quad (58)$$

For every non-zero coupling constant λ , the vacuum expectation value $\langle \phi \rangle$ does not vanish. This is one of the characteristics of the symmetry breaking. Note that not only the scale symmetry of the ground state but also that of the Lagrangian are broken. In fact, the scale symmetry was broken manifestly in the process of doing renormalization: on the onset a mass counterterm has been introduced in the total Lagrangian. Such a symmetry-breaking would result in a trace-anomaly. However, the scale symmetry survives at the false vacuum $\phi = 0$.

Replacing the scaling factor M by $\langle \phi \rangle$, the effective potential can be written as

$$V_1^{VD} = \frac{K\lambda^2\phi^4}{64\pi^2} \left(2 \ln \frac{\phi}{\langle \phi \rangle} - \frac{1}{2} \right) + \Lambda. \quad (59)$$

We would choose

$$\Lambda = \frac{K\lambda^2\langle \phi \rangle^4}{128\pi^2} \quad (60)$$

so that $V_1^{VD}|_{\langle \phi \rangle} = 0$.

Let us consider a cosmological model including the classical matter Lagrangian \mathcal{L}_M and the effective action of the scale invariant gravity:

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{12} \phi^2 R + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V_1^{VD}(\phi) + \mathcal{L}_M \right]. \quad (61)$$

We obtain the field equations,

$$-\frac{\phi^2}{6} \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right) = -\phi^{;\mu} \phi^{;\nu} + \frac{1}{2} g^{\mu\nu} \phi_{;\rho} \phi^{;\rho} - \frac{1}{6} g^{\mu\nu} (\phi^2)_{;\rho}{}^{;\rho} + \frac{1}{6} (\phi^2)^{;\mu}{}^{;\nu} + g^{\mu\nu} V_1^{VD}(\phi) + T_M^{\mu\nu}, \quad (62)$$

$$0 = \phi_{;\mu}{}^{;\mu} + \frac{1}{6} R \phi - V_1^{VD'}(\phi), \quad (63)$$

by varying the action with respect to $g_{\mu\nu}$ and ϕ . Here the stress-energy tensor of classical matter is defined by

$$T_M^{\mu\nu} \equiv 2 \frac{\delta}{\delta g_{\mu\nu}} \int d^4x \sqrt{-g} \mathcal{L}_M. \quad (64)$$

Comparing the covariant derivative of Eq. (62) with Eq. (63), one has the energy-momentum conservation law of the classical matter,

$$T_M^{\mu\nu}{}_{;\nu} = 0. \quad (65)$$

Further, taking trace on Eq. (62) then comparing with Eq. (63), a simple but strong constraint,

$$4V_1^{VD}(\phi) - \phi V_1^{VD'}(\phi) = -T_M \equiv -g_{\mu\nu} T_M^{\mu\nu}. \quad (66)$$

can be obtained. Note that when T_M vanishes, the above equation is exactly the constraint (11) required by the scale symmetry.

In standard cosmology [14], the stress-energy tensor of classical matter can be approximated by a perfect fluid with energy density ρ and pressure p :

$$T^{\mu\nu} = p g^{\mu\nu} + (\rho + p) U^\mu U^\nu, \quad (67)$$

where U^μ is a four-velocity with $U^\mu U_\mu = -1$. Assume a homogeneous and isotropic universe, then ρ , p and ϕ are functions of time only, and the geometry of the universe is given by the Friedmann-Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right], \quad (68)$$

where $a(t)$ is the cosmic scale factor, and $k = 1, 0$ or -1 corresponds to a closed, flat or open universe respectively. Inserting the metric into Eq. (62), one obtains

$$\frac{\phi^2}{2a^2} (\dot{a}^2 + k) = -\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \frac{\dot{a}}{a} (\dot{\phi}^2) + \rho - V_1^{VD}, \quad (69)$$

$$-\frac{\phi^2}{6a^2} (2a\ddot{a} + \dot{a}^2 + k) = -\frac{1}{2} \dot{\phi}^2 + \frac{1}{6} (\ddot{\phi}^2) + \frac{1}{3} \frac{\dot{a}}{a} (\dot{\phi}^2) + p + V_1^{VD}, \quad (70)$$

for $(\mu, \nu) = (0, 0)$ and $(1, 1)$ respectively. And Eqs. (63), (65), and (66) become respectively

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \phi + V_1^{VD'} = 0, \quad (71)$$

$$\frac{dp}{dt} a^3 = \frac{d}{dt} [a^3(\rho + p)], \quad (72)$$

$$\frac{K\lambda^2}{32\pi^2} (\phi^4 - \langle \phi \rangle^4) = 3p - \rho. \quad (73)$$

One can estimate the vacuum expectation value $\langle \phi \rangle$ without knowing the present values of the coupling constant λ , the energy density ρ_0 , and the scale factor a_0 . Since the Newton's constant G is related to the scalar field ϕ by

$$G = \frac{3}{4\pi\phi^2}, \quad (74)$$

it follows that the rate of change of the Newton's constant,

$$\mathcal{G} \equiv \frac{\dot{G}}{G} = -2\frac{\dot{\phi}}{\phi} = -\frac{48\pi^2\rho_0 a_0^3}{K\lambda^2\phi^4 a^3} \frac{\dot{a}}{a} = \frac{3}{2} \left(\frac{\langle \phi \rangle^4}{\phi^4} - 1 \right) H, \quad (75)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. The last two equalities are obtained from Eqs. (72) and (73) by taking $p = 0$ in the current matter-dominated universe.

The present upper bound of the rate of change of G based on the time-delay experiment is [15]

$$|\mathcal{G}_0| \leq \frac{1}{500} H_0. \quad (76)$$

The present value of ϕ is the Planck scale given by

$$\phi_0 = \sqrt{\frac{3}{4\pi G_0}} = 6 \times 10^{18} \text{GeV}. \quad (77)$$

Hence,

$$\left| \frac{\langle \phi \rangle^4 - \phi_0^4}{\phi_0^4} \right| = \frac{2}{3} \frac{|\mathcal{G}_0|}{H_0} \leq \frac{1}{750}, \quad (78)$$

which means that the universe is nearly in the ground state today.

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APPENDIX A: VILKOVISKY-DEWITT METHOD

In this appendix, we would briefly introduce the Vilkovisky-DeWitt method [5, 6, 16, 17] and calculate the one-loop VD effective potential for the scale invariant gravity (28).

Let the naive metric of the space of the configuration of the quantum fields \mathcal{M} be G_{ij} , where i and j are indices which runs over all the quantum fields at every point of the whole space-time. G_{ij} does not have to be gauge invariant.

In the calculation of the one-loop effective potential, one needs to know the second derivative (variation) of the action $S[\phi]$ with respect to the quantum fields. Since S is a scalar on \mathcal{M} , $S_{,i} \equiv \delta S / \delta \phi^i$ is a vector on \mathcal{M} . If G_{ij} describes a non-trivial curved space, one should take the covariant derivative of $S_{,i}$,

$$D_i S_{,j} = \frac{\delta}{\delta \phi^i} S_{,j} - \Gamma_{ij}^k S_{,k}, \quad (A1)$$

instead of $\delta^2 S / \delta \phi^i \delta \phi^j$ because the corresponding effective potential should not depend on the choice of special background field configurations. Here the connection Γ_{ij}^k can be written as

$$\Gamma_{ij}^k = \frac{1}{2} G^{kl} (G_{li,j} + G_{lj,i} - G_{ij,l}), \quad (A2)$$

which is the Christoffel symbol.

In gauge theories, it is possible to define a gauge independent metric from the naive one. Suppose that a general infinitesimal transformation of the fields is a pure gauge transformation, then

$$\delta \phi^i = Q_\alpha^i \epsilon^\alpha, \quad (A3)$$

where Q_α^i is the generator of the gauge symmetry, and ϵ^α is a parameter. In general, $\delta \phi^i$ should include the gauge transformation part and the physical transformation part. The line element of the general transformation $\delta \phi^i$ in \mathcal{M} can be written as

$$\delta s^2 = G_{ij} \delta \phi^i \delta \phi^j. \quad (A4)$$

Define the projection operator as

$$\Pi_j^i \equiv \delta_j^i - Q_\alpha^i N^{\alpha\beta} Q_\beta^k G_{kj}, \quad (A5)$$

satisfying

$$\Pi_j^i Q_\alpha^j = 0, \quad (A6)$$

$$\Pi_j^i \Pi_k^j = \Pi_k^i, \quad (A7)$$

to project out the gauge transformation part of a general infinitesimal transformation of the field. Here $N^{\alpha\beta}$ is the inverse of

$$N_{\alpha\beta} = G_{ij} Q_{\alpha}^i Q_{\beta}^j. \quad (\text{A8})$$

The component of $\delta\phi^i$ in the physical space can then be defined by

$$\delta_{\perp}\phi^i \equiv \Pi_j^i \delta\phi^j, \quad (\text{A9})$$

and the line element of the physical transformation reads

$$\delta_{\perp}s^2 = G_{ij} \delta_{\perp}\phi^i \delta_{\perp}\phi^j \equiv \gamma_{ij} \delta\phi^i \delta\phi^j, \quad (\text{A10})$$

where

$$\gamma_{ij} = G_{ik} \Pi_j^k \quad (\text{A11})$$

is taken to be the gauge independent metric which measures the physical transformation part of a general transformation. Using γ_{ij} , the connection in the usual definition can be constructed in terms of G_{ij} and Q_{α}^i as

$$\Gamma^{(\gamma)}_{ij}{}^k = \Gamma_{ij}^k + T_{ij}^k, \quad (\text{A12})$$

where

$$T_{ij}^k = -2Q_{\alpha;(i}^k B_{j)}^{\alpha} + Q_{\sigma}^l B_{(i}^{\sigma} B_{j)}^{\rho} Q_{\rho;l}^k \quad (\text{A13})$$

with

$$B_i^{\alpha} \equiv N^{\alpha\beta} Q_{\beta}^i G_{ij}, \quad (\text{A14})$$

$$Q_{\alpha;i}^k \equiv \frac{\delta}{\delta\phi^i} Q_{\alpha}^k + \Gamma_{ij}^k Q_{\alpha}^j. \quad (\text{A15})$$

Here the convention of symmetrization,

$$A_{(i} B_{j)} \equiv \frac{1}{2} (A_i B_j + A_j B_i), \quad (\text{A16})$$

is understood. Now the covariant derivative of S_i with the connection $\Gamma^{(\gamma)}_{ij}{}^k$ is gauge independent, so does the effective potential V_1^{VD} constructed from it.

Now we turn to our case of the scale invariant gravity (28). Following Vilkovisky's prescription [5], we take the naive metric,

$$G_{\tilde{\phi}(x)\tilde{\phi}(y)} = \sqrt{-g} \delta(x-y), \quad (\text{A17})$$

$$G_{g_{\mu\nu}(x)g_{\rho\sigma}(y)} = \frac{1}{2} \sqrt{-g} (g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho} - g^{\mu\nu} g^{\rho\sigma}) \delta(x-y), \quad (\text{A18})$$

and the infinitesimal transformations,

$$\delta\tilde{\phi} = -\epsilon^{\alpha} \partial_{\alpha} \tilde{\phi} - \omega \tilde{\phi} \equiv Q_{\epsilon^{\alpha}}^{\tilde{\phi}} \epsilon^{\alpha} + Q_{\omega}^{\tilde{\phi}} \omega, \quad (\text{A19})$$

$$\delta g_{\mu\nu} = -g_{\alpha\mu} \partial_{\nu} \epsilon^{\alpha} - g_{\alpha\nu} \partial_{\mu} \epsilon^{\alpha} - \epsilon^{\alpha} \partial_{\alpha} g_{\mu\nu} + 2\omega g_{\mu\nu} \equiv Q_{\epsilon^{\alpha}}^{g_{\mu\nu}} \epsilon^{\alpha} + Q_{\omega}^{g_{\mu\nu}} \omega, \quad (\text{A20})$$

where ϵ^{α} and ω are parameters corresponding to the reparametrization and scale transformations respectively. Then the generators of these transformations are

$$Q_{\epsilon^{\alpha}(x)}^{\tilde{\phi}(z)} = -\partial_{\alpha} \tilde{\phi} \delta(z-x), \quad (\text{A21})$$

$$Q_{\epsilon^{\alpha}(x)}^{g_{\mu\nu}(z)} = -g_{\alpha\mu} \partial_{\nu} \delta(z-x) - g_{\alpha\nu} \partial_{\mu} \delta(z-x) - \delta(z-x) \partial_{\alpha} g_{\mu\nu}(z), \quad (\text{A22})$$

$$Q_{\omega(x)}^{\tilde{\phi}(z)} = -\tilde{\phi}(z) \delta(z-x), \quad (\text{A23})$$

$$Q_{\omega(x)}^{g_{\mu\nu}(z)} = 2g_{\mu\nu}(z) \delta(z-x). \quad (\text{A24})$$

Here the derivative ∂_μ always acts on the first argument of the δ function.

A straightforward calculation gives the quantities (A8) evaluated at the ground-state background values (33) and (34),

$$N_{\epsilon^\alpha(x)\epsilon^\beta(y)}|_{bg} = -2\eta_{\alpha\beta}\partial^2\delta(x-y), \quad (A25)$$

$$N_{\omega(x)\omega(y)}|_{bg} = -15\delta(x-y), \quad (A26)$$

$$N_{\epsilon^\alpha(x)\omega(y)}|_{bg} = -N_{\omega(y)\epsilon^\alpha(x)}|_{bg} = -4\partial_\alpha\delta(x-y), \quad (A27)$$

whose inverses read

$$N^{\epsilon^\alpha(x)\epsilon^\beta(y)}|_{bg} = \left(-\frac{1}{2}\eta^{\alpha\beta}\frac{1}{\partial^2} - \frac{4}{23}\frac{\partial^\alpha\partial^\beta}{\partial^4}\right)\delta(x-y), \quad (A28)$$

$$N^{\omega(x)\omega(y)}|_{bg} = -\frac{1}{23}\delta(x-y), \quad (A29)$$

$$N^{\epsilon^\alpha(x)\omega(y)}|_{bg} = -N^{\omega(x)\epsilon^\alpha(y)}|_{bg} = \frac{2}{23}\frac{\partial^\alpha}{\partial^2}\delta(x-y). \quad (A30)$$

Hence, the background values of the quantities defined in Eq. (A14) are

$$B_{\tilde{\phi}(z)}^{\epsilon^\alpha(u)}|_{bg} = -\frac{2}{23}\frac{\partial^\alpha}{\partial^2}\delta(u-z), \quad (A31)$$

$$B_{\tilde{\phi}(z)}^{\omega(u)}|_{bg} = \frac{1}{23}\delta(u-z), \quad (A32)$$

$$B_{g_{\mu\nu}(z)}^{\epsilon^\alpha(u)}|_{bg} = \left(-\eta^{\alpha(\mu}\frac{\partial^{\nu)}}{\partial^2} + \frac{7}{46}\eta^{\mu\nu}\frac{\partial^\alpha}{\partial^2} + \frac{8}{23}\frac{\partial^\alpha\partial^\mu\partial^\nu}{\partial^4}\right)\delta(u-z), \quad (A33)$$

$$B_{g_{\mu\nu}(z)}^{\omega(u)}|_{bg} = -\frac{4}{23}\left(\frac{\partial^\mu\partial^\nu}{\partial^2} - \eta^{\mu\nu}\right)\delta(u-z). \quad (A34)$$

It follows that T_{ij}^k in Eq. (A13) are given by

$$T_{\tilde{\phi}(x)\tilde{\phi}(y)}^{\tilde{\phi}(z)} = \frac{-3}{529}\delta(x-z)\delta(y-z), \quad (A35)$$

$$\eta^{\rho\sigma}T_{\tilde{\phi}(x)\tilde{\phi}(y)}^{g_{\rho\sigma}(z)} = \frac{71}{529}\delta(x-z)\delta(y-z), \quad (A36)$$

$$T_{g_{\mu\nu}(x)\tilde{\phi}(y)}^{\tilde{\phi}(z)} = \left(\frac{553}{1058}\frac{\partial^\mu\partial^\nu}{\partial^2} + \frac{277}{1058}\eta^{\mu\nu}\right)\delta(x-z)\delta(y-z), \quad (A37)$$

$$\eta^{\rho\sigma}T_{g_{\mu\nu}(x)\tilde{\phi}(y)}^{g_{\rho\sigma}(z)} = \left(-\frac{192}{529}\frac{\partial^\mu\partial^\nu}{\partial^2} + \frac{146}{529}\eta^{\mu\nu}\right)\delta(x-z)\delta(y-z), \quad (A38)$$

$$T_{g_{\alpha\beta}(x)g_{\mu\nu}(y)}^{\tilde{\phi}(z)} = \left[-\frac{48}{529}\frac{\partial^\alpha\partial^\beta\partial^\mu\partial^\nu}{\partial^4} + \frac{25}{529}\left(\eta^{\mu\nu}\frac{\partial^\alpha\partial^\beta}{\partial^2} + \eta^{\alpha\beta}\frac{\partial^\mu\partial^\nu}{\partial^2}\right) - \frac{2}{529}\eta^{\alpha\beta}\eta^{\mu\nu}\right]\delta(x-z)\delta(y-z), \quad (A39)$$

$$\eta^{\rho\sigma}T_{g_{\alpha\beta}(x)g_{\mu\nu}(y)}^{g_{\rho\sigma}(z)} = \left[\frac{400}{529}\frac{\partial^\alpha\partial^\beta\partial^\mu\partial^\nu}{\partial^2} - 2\frac{\partial^{(\mu}\eta^{\nu)(\alpha}\partial^{\beta)}}{\partial^2} + \frac{313}{529}\left(\eta^{\mu\nu}\frac{\partial^\alpha\partial^\beta}{\partial^2} + \eta^{\alpha\beta}\frac{\partial^\mu\partial^\nu}{\partial^2}\right) - \frac{465}{1058}\eta^{\alpha\beta}\eta^{\mu\nu}\right]\delta(x-z)\delta(y-z). \quad (A40)$$

On the other hand, the background values of the connection of the naive metric (A17) and (A18) are

$$\Gamma_{\tilde{\phi}(x)g_{\mu\nu}(y)}^{\tilde{\phi}(z)}|_{bg} = \Gamma_{g_{\mu\nu}(y)\tilde{\phi}(x)}^{\tilde{\phi}(z)}|_{bg} = \frac{1}{4}\eta^{\mu\nu}\delta(x-z)\delta(y-z), \quad (A41)$$

$$\Gamma_{\tilde{\phi}(x)\tilde{\phi}(y)}^{g_{\mu\nu}(z)}|_{bg} = \frac{1}{4}\eta_{\mu\nu}\delta(x-z)\delta(y-z), \quad (A42)$$

$$\begin{aligned} \Gamma_{g_{\mu\nu}(x)g_{\alpha\beta}(y)}^{g_{\rho\sigma}(z)}|_{bg} = & -\frac{1}{2}\delta(x-z)\delta(y-z)\left[\eta^{\alpha(\mu}\delta_{(\rho}^{\nu)}\delta_{\sigma)}^{\beta)} + \eta^{\beta(\mu}\delta_{(\rho}^{\nu)}\delta_{\sigma)}^{\alpha)}\right. \\ & \left.-\frac{1}{2}\eta^{\alpha\beta}\delta_{(\rho}^{\mu}\delta_{\sigma)}^{\nu)} - \frac{1}{2}\eta^{\mu\nu}\delta_{(\rho}^{\alpha}\delta_{\sigma)}^{\beta)} - \frac{1}{2}\eta_{\rho\sigma}\eta^{\alpha(\mu}\eta^{\nu)\beta)} + \frac{1}{4}\eta_{\rho\sigma}\eta^{\alpha\beta}\eta^{\mu\nu}\right]. \end{aligned} \quad (A43)$$

Combining these with T_{ij}^k , the gauge independent connection (A12) and hence the covariant derivative (A1) can be worked out. This is equivalent to adding the correction terms,

$$\begin{aligned}\mathcal{L}' = & \frac{1}{2}\sigma\left(-\frac{12}{529}\lambda\phi^2\right)\sigma + \sigma\left(-\frac{505}{6348}\lambda\phi^2\right)\frac{\partial^\mu\partial^\nu}{\partial^2}\sigma + \sigma\left(-\frac{2}{529}\lambda\phi^2\right)h + \frac{1}{2}h\left(\frac{497}{50784}\lambda\phi^2\right)h \\ & + \frac{1}{2}h_{\mu\nu}\left(-\frac{\lambda\phi^2}{1587}\right)\frac{\partial^\mu\partial^\nu\partial^\alpha\partial^\beta}{\partial^4}h_{\alpha\beta} + \frac{1}{2}h_{\mu\alpha}\left(\frac{1}{24}\lambda\phi^2\right)\frac{\partial^\alpha\partial_\beta}{\partial^2}h^{\mu\beta} + h\left(-\frac{171}{8464}\lambda\phi^2\right)\frac{\partial^\mu\partial^\nu}{\partial^2}h_{\mu\nu},\end{aligned}\quad (\text{A44})$$

to the original quadratic Lagrangian (49).

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